Equality of Dedekind sums: experimental data and theory

Challenges in 21st Century
Experimental Mathematical Computation

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What are Dedekind sums?

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1. They are natural extensions of the *gcd* function.
2. They give the characters of $SL_2(Z)$ (via the Rademacher function).
3. They are the building blocks for integer point enumeration in polytopes.
4. They are the variance of congruential pseudo-random number generators.
5. They are necessary in the transformation law of Dedekind's eta function.
6. They give the linking number between some knots
7. They provide correction terms for the Heegaard-Floer homology

Definition.

We define the first periodic Bernoulli polynomial by

$$B(x) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

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Another expression for the Dedekind sum, as an infinite series, is somewhat better:

$$\left(-\frac{4\pi^2}{b}\right) s(a,b) = \sum\nolimits'_{(m,n)\in\mathbb{Z}^2} \frac{1}{m(am+bn)},$$

where the dash in the summation denotes omission of the two discrete lines $m = 0$ and $am + bn = 0$.

This representation gives an easy proof of the important RECIPROCITY LAW FOR DEDEKIND SUMS:

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For any two relatively prime integers a and b, we have

$$\boxed{s(a,b) + s(b,a) = \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) - \frac{1}{4}.}$$

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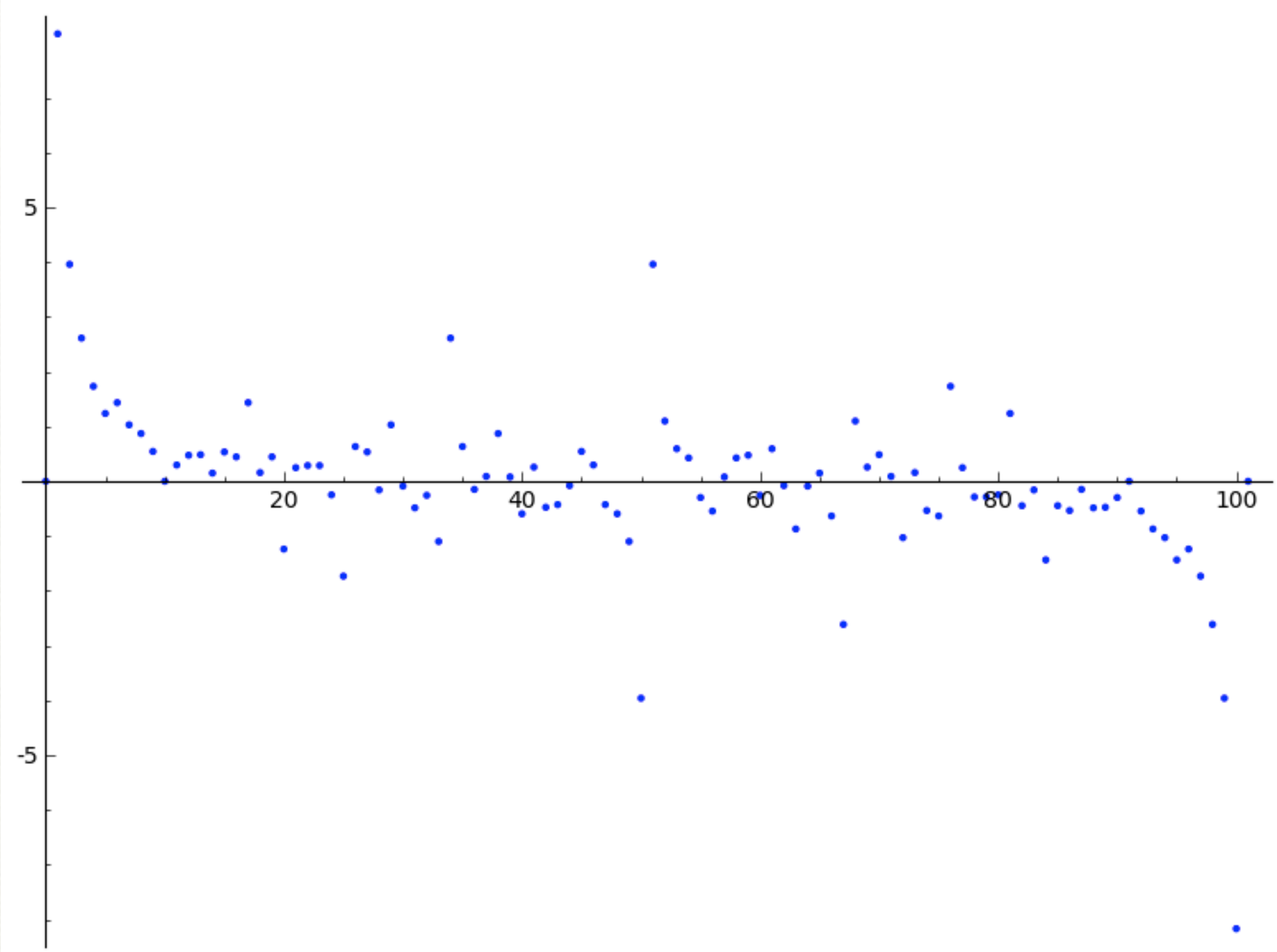
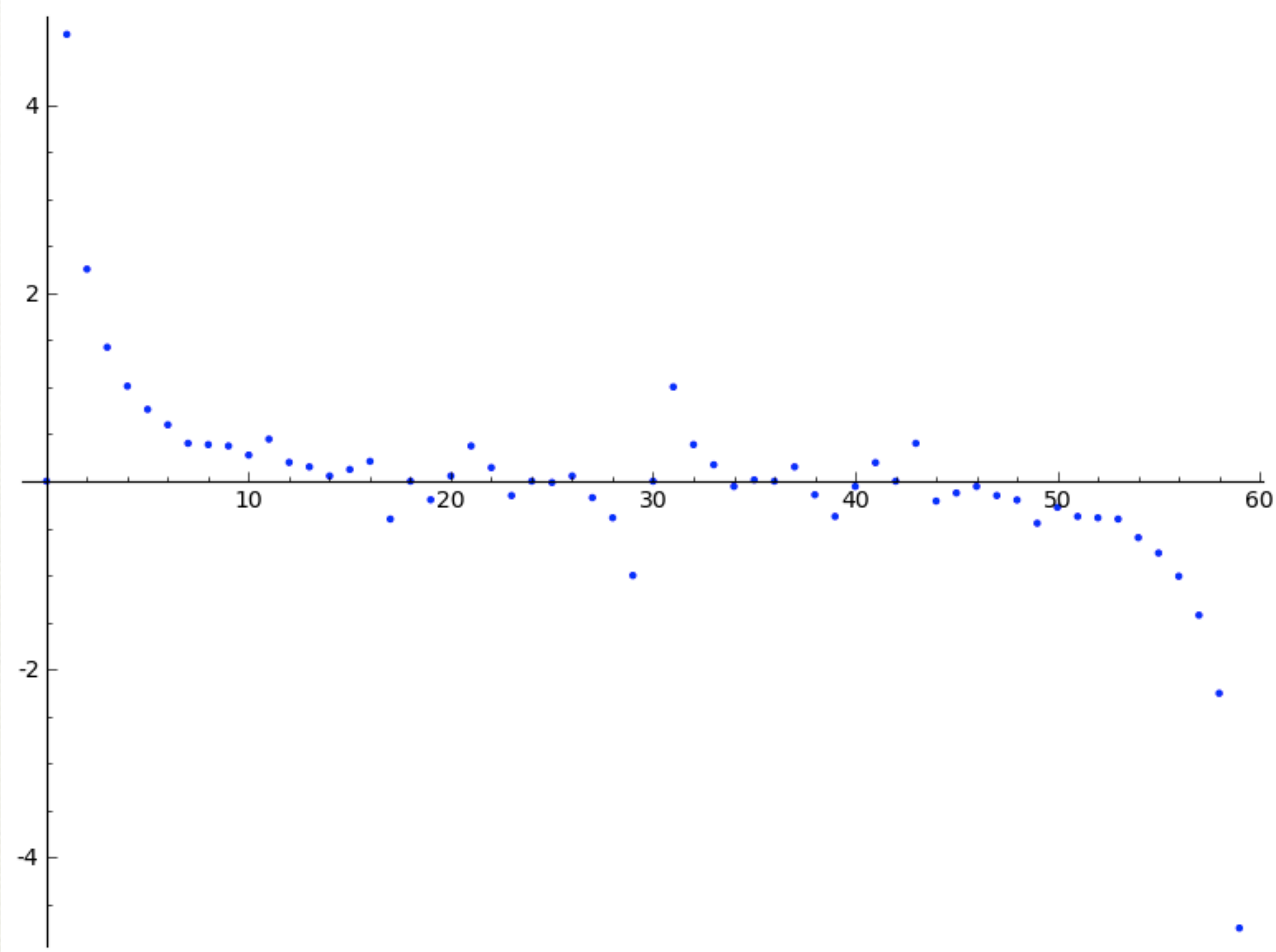
Thus, we see that this classical Dedekind sum behaves precisely like the gcd function, as far as computational complexity.



Plot of $(a,\ s(a, 101))$, for $a = 0, \ldots, 101$.



"smoother?"

Plot of $(a,\ s(a,60))$, for $a = 0, \ldots, 60$.

Integer point enumeration in polytopes

More generally, we have the Eisenstein-Dedekind sums, defined by:

$$s(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_d}) := \frac{|\det V|}{(2\pi i)^d} \sum\nolimits'_{\mathbf{m} \in \mathbb{Z}^{\mathbf{d}}} \frac{e^{2\pi i \langle \mathbf{m}, \mathbf{u} \rangle}}{\Pi_{k=1}^{d} \langle \mathbf{v_k}, \mathbf{m} \rangle},$$

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- Paul Gunnells and Robert Sczech, Evaluation of Dedekind sums, Eisenstein cocycles, and special values of L-functions, Duke Math. J. **118** (2003), no. 2, 229–260.

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Theorem. (Ehrhart, 1957)

$|tP \cap \mathbb{Z}^d|$ is a polynomial in $t \in \mathbb{Z}_{>0}$, given by

$$|tP \cap \mathbb{Z}^d| = vol(\mathcal{P})t^d + c_{d-1}t^{d-1} + \cdots + c_1 t + 1.$$

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The coefficients c_j have as their building blocks the Dedekind sums and their higher-dimensional analogues.

Pommersheim, James E. Toric varieties, lattice points and Dedekind sums. *Math. Ann.* 295 (1993), no. 1, 1–24.

Diaz, Ricardo; Robins, Sinai The Ehrhart polynomial of a lattice polytope. *Ann. of Math. (2)* 145 (1997),no. 3, 503–518.

Characters of $SL_2(\mathbb{Z})$ and the Rademacher function

Fix any matrix $M := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

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$$R(M) := \begin{cases} \frac{a+d}{c} - 12(\text{sign } c)s(d, |c|) & \text{for } c \neq 0, \\ \frac{b}{d} & \text{for } c = 0. \end{cases}$$

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It is a fact that R maps $SL_2(Z)$ into the integers, and that furthermore given any three unimodular matrices $M_1, M_2, M_3 \in SL_2(Z)$ which enjoy the relation $M_3 := M_1 M_2$, we have

$$R(M_3) = R(M_1) + R(M_2) - 3 \text{ sign}(c_1 c_2 c_3).$$

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and it is known that all the characters of $SL_2(Z)$ may be obtained in this manner, forming the group $\mathbb{Z}/12\mathbb{Z}$.

(The commutator subgroup of $SL_2(\mathbb{Z})$ has index 12 in $SL_2(\mathbb{Z})$)

Modular forms: the Dedekind Eta function

For each τ in the complex upper half plane, the Dedekind η-function is defined by:

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q := e^{2\pi i \tau}$.

Richard Dedekind (1892) proved the general transformation law under any element $M := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$:

Modular forms: the Dedekind Eta function

Theorem (R. Dedekind)

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) := e^{\frac{2\pi i R(M)}{24}}\sqrt{\frac{c\tau+d}{i}}\,\eta(\tau),$$

for all τ in the complex upper half plane.

We also see the appearance of a character in this transformation law.

Surprisingly many modular forms may be built up by taking products and quotients of this function.

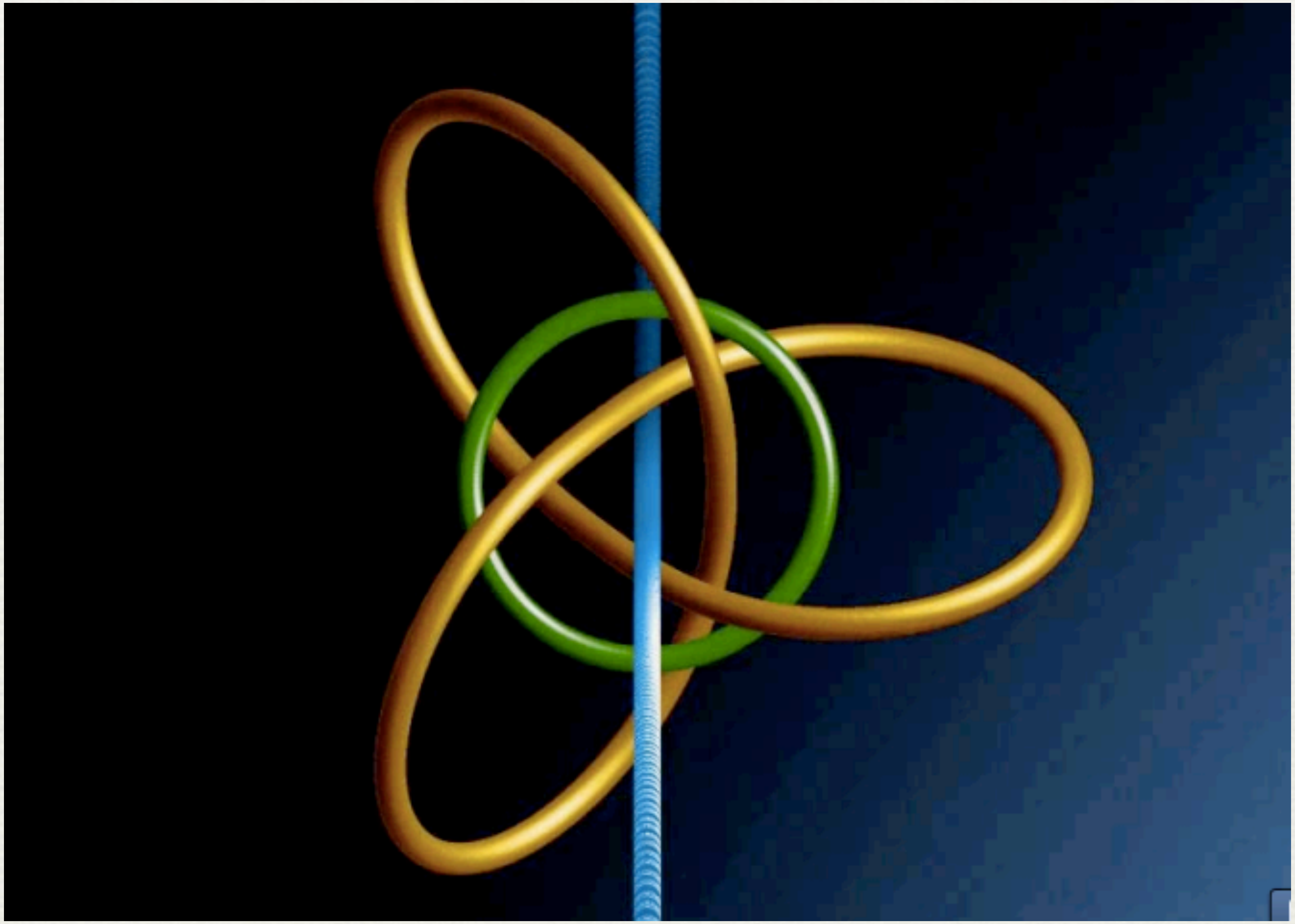
The Linking Number of knots

Each hyperbolic matrix $M \in PSL_2(\mathbb{Z})$ defines a closed curve k_M in the complement of the trefoil knot.

Theorem. (*Étienne Ghys*)

The linking number between k_M and the trefoil knot is equal to the Rademacher function $R(M)$.

The Linking Number of knots



Ref: http://perso.ens-lyon.fr/ghys/articles/icm.pdf

When are two Dedekind sums equal?

Open Problem.

We are interested in the question of finding all integers $1 \leq a_1, a_2 \leq b-1$ for which

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Theorem. (Jabuka, Robins, Wang, 2011) Let b be a positive integer, and a_1, a_2 any two integers that are relatively prime to b. If $s(a_1, b) = s(a_2, b)$, then

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Jabuka, Stanislav; Robins, Sinai; Wang, Xinli When are two Dedekind sums equal?
Int. J. Number Theory 7 (2011), no. 8, 2197–2202.

Jabuka, Stanislav; Robins, Sinai; Wang, Xinli Heegaard Floer correction terms and Dedekind-Rademacher sums.
Int. Math. Res. Not. IMRN 2013, no. 1, 170–183. 57R58

Girstmair, Kurt A criterion for the equality of Dedekind sums mod Z, *Int. J. Number Theory*
10 (2014),no. 3, 565–568.

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As a corollary, when $b = p$ is a prime, we have that $s(a_1, p) = s(a_2, p)$ if and only if

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Open problem (special case). For $b = pq$, where p, q are primes, find all pairs of integers a_1, a_2 such that $s(a_1, pq) = s(a_2, pq)$.

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Suppose we wish to study the number of inversions of the following permutation. For each a relatively prime to b, we define the permutation:

$$\sigma_a := \begin{pmatrix} 1 & 2 & 3 & \cdots & b-1 \\ a & 2a & 3a & \cdots & (b-1)a \end{pmatrix}$$

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For each such permutation σ_a, we let $Inv(\sigma_a)$ be the number of inversions of the permutation σ_a, i.e. the number of times that a larger integer precedes a smaller integer in this permutation.

Theorem. (Zolotareff, 1872)

For each a relatively prime to b, we have

$$Inv(\sigma_a) = -3b\ s(a,b) + \frac{1}{4}(b-1)(b-2).$$

Rademacher, Hans; Grosswald, Emil Dedekind sums. The Carus Mathematical Monographs, No. 16.*The Mathematical Association of America, Washington, D.C.,* 1972. xvi+102 pp.

64. E. Zolotareff, Nouvelle démonstration de la loi de réciprocité de Legendre, Nouvelles Annales de Math., (2), 11 (1872) 355–362.

Another motivation is the distribution of the difference between a unit mod b and its inverse mod b:

$$a - a^{-1} \equiv 12b\ s(a,b) \mod b,$$

where a^{-1} is the inverse of $a \mod b$.

Empirically, we can see equality of Dedekind sums $S(a_i, b)$ for long arithmetic progressions of $a_i \mod b$, some of which is currently provable, and some of which is currently conjectural.

Thank You